Technical Appendix for Dollar Illiquidity and Central Bank Facilities During the Global Financial Crisis

Andrew K. Rose

Mark M. Spiegel*

August 2, 2011

1 Setup

There are two countries in the model,labeled u and r, which can be interpreted as representing the United States and the rest of the world. To keep the analysis as simple as possible, we assume that their characteristics are identical, except where indicated. In particular, we assume that country z has an overall output share of $\tau_z; (z = u, r)$, where $0 \leq \tau_z \leq 1$ and $\tau_u = 1 - \tau_r$.

In each period in each country, a continuum of infinitely lived agents participate in two distinct international markets: One is a Walrasian centralized global market, and another is a decentralized market, where pairs of buyers and sellers from the two countries are randomly matched. Transactions in the decentralized market are characterized by a double-coincidence problem, which rules out barter, and anonymity, which rules out the provision of credit between matched agents. It therefore follows that a tangible medium of exchange is required for transactions to take place in the decentralized market.[1]

*This appendix derives the results tested in Dollar Illiquidity and Central Bank Facilities During the Global Financial Crisis, by Andrew K. Rose and Mark M. Spiegel. All views presented are those of authors and do not represent the views of the Federal Reserve Bank of San Francisco or Federal Reserve System.

[1]These assumptions follow directly from Lagos and Wright (2005). As in that paper, the assumption of no barter

Preferences and production technologies are assumed to be identical across countries. On each date, agents from country z ($z = u, r$) can produce a tradable homogeneous good for the centralized market, x, using labor, h_z, according to the production function $x_z = h_z$. The law of one price holds in this market. Utility is assumed to be concave in x and negatively linear in h according to $U(x_z) - h_z$ and $U'(0) = \infty$, so that x_z^*, the optimal production of x in each country satisfies $U(x_z^*) = 1$.

Agents also produce a good, q_z, which is tradable in the international decentralized market. q_z is produced at disutility $c(q_z)$, where $c' > 0$, $c" > 0$, and $c(0) = c'(0) = 0$. Agents value q_z according to the concave function $\upsilon(q_z)$, where $\upsilon' > 0$, $\upsilon" < 0$, $\upsilon(0) = 0$, and $\upsilon'(0) = \infty$, so that q_z^*, the optimal production of q_z satisfies $\upsilon'(q_z) = c'(q_z)$. To highlight the role that differences in information sets and asset illiquidity play in determining outcomes, we assume that both x and q are homogeneous across countries.

There are four assets in the model. Each economy has a domestic money supply, discussed in more detail below, as well as a real asset, which is like a Lucas tree. All agents have perfect information about the value of their economy's money, which is in fixed supply. The real assets yield a dividend in the centralized market the following period. There are good assets and bad assets. Bad assets yield a zero dividend, while good assets yield a dividend of δ_z units of x; $z = u, r$. Moreover, unlike money, bad assets can be produced by sellers at zero cost.

As in Lester, Postlewaite, and Wright (2009), all agents can distinguish between bad and good assets in the centralized market, but in the decentralized market only informed agents can make this distinction. Since bad assets can be produced at zero cost, sellers who do not know the value of an underlying asset will refuse to accept it at a positive price. This yields the simplification that bargaining only takes place under situations where both agents are informed. Finally, note that money can have value, although it also yields zero dividends, because it is in fixed supply and

and credit is stronger than necessary and only maintained for simplicity. It is not necessary that barter and credit are ruled out for all transactions in the decentralized market, only a portion of them.

provides liquidity services. Let ϕ_z and ψ_z represent the values of money and real assets of country z in the centralized market in terms of x respectively.

We focus on steady state equilibria. There is a fixed supply of trees in each country, A_z, and the supplies of both currencies grow at a constant rate, γ_z. Let $\widehat{k}$ represent the next period value of any variable k, so that $\widehat{M_z} = \gamma_z M_z$. Agents worldwide are assumed to share a common discount factor, β, and we assume that $\gamma_z > \beta$ for both countries.

It has been shown [e.g. Lagos and Rocheteau (2008)] that agents may choose to keep some of their assets out of the bargaining process in the decentralized market if they are allowed to do so, as the endowments of each agent can affect the bargaining outcome. This would be true in our model as well. However, to accommodate assets from two countries without too much complexity, we make the simplifying assumption that all assets owned by agents are brought into the decentralized market. We also assume that assets are "scarce," and therefore carry a liquidity value over their value in exchange the following day in the centralized market. The conditions needed for this assumption to hold are shown below.

2 Centralized market

Agents from country z ($z = u, r$) choose a portfolio comprised of four assets: $m_{z,u}$ units of country u currency, $m_{z,r}$ units of country r currency, $a_{z,u}$ units of country u real assets, and $a_{z,r}$ units of country r assets. Let y_z represent income of an agent from country z in the centralized market, which satisfies

$$y_z = \phi_u m_{z,u} + \phi_r m_{z,r} + (\delta_u + \psi_r) a_{z,u} + (\delta_r + \psi_r) a_{z,r}. \tag{1}$$

Let $W(y_r)$ be the value function of an agent from country z in the centralized market, and define $V_z(m_{z,u}, m_{z,r}, a_{z,u}, a_{z,r})$ as the value function of an agent from country z in the decentralized market with portfolio $(m_{z,u}, m_{z,r}, a_{z,u}, a_{z,r})$. The optimization problem in the centralized market for an agent from country z then satisfies

$$\max_{x_z, h_z, \widehat{m}_{z,u}, \widehat{m}_{z,r}, \widehat{a}_{z,u}, \widehat{a}_{z,r}} W(y_z) = \{U(x_z) - h_z + \beta V_{z,u}(\widehat{m}_{z,u}, \widehat{m}_{z,r}, \widehat{a}_{z,u}, \widehat{a}_{z,r})\} \tag{2}$$

subject to

$$x_z \leq h_z + y_z - \phi_u \widehat{m}_{z,u} - \phi_r \widehat{m}_{z,r} - \psi_u(\widehat{a}_{z,u}) - \psi_r(\widehat{a}_{z,r}) + T_z, \tag{3}$$

where T_z is a lump-sum transfer returned to private agents in country z from revenues generated by money creation, $T_z = (\gamma_z - 1)M_z$. Finally, we assume that $\gamma_z > 1$ and as in Lagos and Wright (2005), we assume that any constraints on h_z, $h_z \epsilon \overline{h}$ are not binding.

Agents' first order conditions satisfy

$$U'(x_z) = 1, \tag{4}$$

$$\phi_u \geq \beta \frac{\partial V_z}{\partial \widehat{m}_{z,u}}, \tag{5}$$

$$\phi_r \geq \beta \frac{\partial V_z}{\partial \widehat{m}_{z,r}}, \tag{6}$$

$$\psi_u \geq \beta \frac{\partial V_z}{\partial \widehat{a}_{z,u}}, \tag{7}$$

and

$$\psi_r \geq \beta \frac{\partial V_z}{\partial \hat{a}_{z,r}}. \tag{8}$$

where the latter four conditions hold with equality when $m_{z,u}$, $m_{z,r}$, $a_{z,u}$, and $a_{z,r}$ are strictly positive, respectively. Note that y_z does not enter into the first order conditions and $W'(y_z) = 1$. This is the mechanism through which the degenerate portfolio solutions are recovered each time the agents return to the centralized market in the Lagos and Wright (2005) framework.

Finally, there are four asset market clearing conditions, as the representative agent from each country holds his country's share of each asset:

$$M_u = m_{u,u} + m_{r,u}, \tag{9}$$

$$M_r = m_{u,r} + m_{r,r}, \tag{10}$$

$$A_u = a_{u,u} + a_{u,r}, \tag{11}$$

and

$$A_r = a_{u,r} + a_{r,r}. \tag{12}$$

3 Decentralized market

We next turn to the equilibrium in the decentralized market. In the decentralized market, agents are randomly paired into bilateral meetings. Let z and k represent the countries of origin of the buyer and seller respectively in the decentralized market ($z, k = u, r$). Buyers can be paired with sellers from their own country $z = k$, or with sellers from the foreign country $z \neq k$. To highlight the possibility of liquidity differences arising across countries, we assume that sellers in the decentralized market only accept assets denominated in their domestic currencies in exchange.[2]

The probability of an agent from country z being paired with an agent from country k with a coincidence of wants is exogenous, and proportional to the share of output of country k, τ_k. In addition, we assume that the probability of a coincidence of wants is greater among agents originating from the same country by an exogenous parameter α, where $\alpha > 1$.

Specifically, let $\lambda_{z,k}$ represent the chance of an agent from country z being paired with an agent from country k from whom he would want to buy, and $\widetilde{\lambda}_{z,k}$ represent the chance of an agent from country z being paired in a meeting with an agent from country k to whom he wants to sell. We assume that $\lambda_{z,k} \equiv \lambda\tau_k$ when $z \neq k$ and $\lambda_{z,k} \equiv \lambda\alpha\tau_k$ when $z = k$, where λ is an exogenous constant term. Similarly, we assume that $\widetilde{\lambda}_{z,k} \equiv \widetilde{\lambda}\tau_k$ when $z \neq k$ and $\widetilde{\lambda}_{z,k} \equiv \widetilde{\lambda}\alpha\tau_k$ when $z = k$, where $\widetilde{\lambda}$ is an exogenous constant term.

Outcomes in the decentralized market are a function of the portfolio of assets held by the buyer as well as the seller's information set. We assume that all agents from country k are fully informed about the value of their domestic currency, m_k ($k = u, r$). However, we assume that only a fraction of agents in country k, ρ_k, are informed about the value of the opaque asset a_k, where $0 \leq \rho_k \leq 1$. ρ_k is therefore also the probability that a randomly selected seller from k is willing to accept both m_k and a_k in transactions, while $1 - \rho_k$ represents the probability that a seller from country k is uninformed about the value of a_k and is only willing to accept m_k as payment. As in Lester, Postlewaite, and Wright (2009), let meetings where the seller is informed about a_k be called "type 2," and meetings where the seller is uninformed be called "type 1." The type of meeting that

[2]This assumption is made for tractability. In practice, the qualitative results would go through with assets from the other country being subject to increased transactions costs. This assumption serves to simplify the decision rule, as we only need to consider two types of agents from each country, informed and uninformed.

is taking place is known to all.

We next examine the characteristics of a type n meeting ($n = 1, 2$) where there is a coincidence of wants between a buyer from country z and a seller from country k. Let $p_{z,k,n}$ represent the price paid by the buyer from country z to a seller from country k for $q_{z,k,n}$ units of the good in a type n meeting. Let $(m_{z,u}, m_{z,r}, a_{z,u}, a_{z,r})$ represent the buyer's portfolio, and $(\widetilde{m}_{k,u}, \widetilde{m}_{k,r}, \widetilde{a}_{k,u}, \widetilde{a}_{k,r})$ represent the seller's portfolio, and y_z and y_k represent the wealth of the buyer and the seller respectively. Finally, let $\omega_{z,k,n}$ be the value of acceptable funds possessed by the buyer, i.e. those recognized by the seller and denominated in the seller's domestic currency. Given our assumptions above, $\omega_{z,k,1} = \phi_k m_{z,k}$, and $\omega_{z,k,2} = \phi_k m_{z,k} + (\psi_k + \delta_k) a_{z,k}$.

Assuming that the buyer has bargaining power θ and threat points are given by continuation values, the generalized Nash bargaining solution is similar to that in Lagos and Wright (2005):[3]

$$\max_{q_{z,k,n}, p_{z,k,n}} [[\upsilon(q_{z,k,n}) + W(y_z - p_{z,k,n})] - W_z(y_z)]^{\theta}[[-c(q_{z,k,n}) + W(y_k + p_{z,k,n})] - W(y_k)]^{1-\theta} \quad (13)$$

subject to $p_{z,k,n} \leq \omega_{z,k,n}$.

The first order conditions satisfy

$$p_{z,k,n} = \frac{\theta \upsilon'(q_{z,k,n}) c(q_{z,k,n}) + (1-\theta)\upsilon(q_{z,k,n}) c'(q_{z,k,n})}{\theta \upsilon'(q_{z,k,n}) + (1-\theta) c'(q_{z,k,n})} \equiv \eta(q_{z,k,n}), \quad (14)$$

and

[3] The generalized bargaining solution is based on the assumption that the alternative to the bargaining outcome is autarky. We give buyers from either country identical bargaining power, θ, for simplicity.

$$-\theta[-c(q_{z,k,n})+p_{z,k,n}]+(1-\theta)[\upsilon(q_{z,k,n})-p_{z,k,n}]-\varphi[-c(q_{z,k,n})+p_{z,k,n}]^{\theta}[\upsilon(q_{z,k,n})-p_{z,k,n}]^{(1-\theta)}=0. \tag{15}$$

We assume that we are in the case where the liquidity constraint is binding, which implies that $p_{z,k,n} = \omega_{z,k,n}$ and $q_{z,k,n}$ satisfies 14 for $p_{z,k,n} = \omega_{z,k,n}$. Note that the terms of trade only depend on the buyer's portfolio.

The value function of an agent from country z in the decentralized market is then equal to the probabilities of being a buyer in a type 1 or 2 meeting with a seller from county k, times the payoffs in those meetings, plus the probability of being either a seller or in a meeting with no opportunity for trade, plus a constant term, Ψ_z.

$$V_z = \sum_{n=1}^{2}[\lambda_{u,n}[\upsilon(q_{z,u,n}) + W(y_z - p_{z,u,n})] + \lambda_{r,n}[\upsilon(q_{z,r,n}) + W(y_z - p_{z,r,n})]] + (1-\lambda)W(y_z) + \Psi_k \tag{16}$$

where $\lambda_{k,1} = \lambda_k(1-\rho_k), \lambda_{k,2} = \lambda_k\rho_k$, and Ψ_k represents the extra utility of an agent from country k associated with being a seller relative to having no trade opportunities.

To solve for Ψ_k, let $\widetilde{q}_{z,k,n}$ and $\widetilde{p}_{z,k,n}$ represent the volume of q sold to an agent from country z, and the proceeds of the sale respectively. Ψ_k satisfies

$$\Psi_k = \{\widetilde{\lambda}_i[-c(\widetilde{q}_{i,k,1})+\widetilde{p}_{i,k,1}]+\widetilde{\lambda}_j[-c(\widetilde{q}_{j,k,1})+\widetilde{p}_{j,k,1}]\}(1-\Phi_k)+\{\widetilde{\lambda}_i[-c(\widetilde{q}_{i,k,2})+\widetilde{p}_{i,k,2}]+\widetilde{\lambda}_j[-c(\widetilde{q}_{j,k,2})+\widetilde{p}_{j,k,2}]\}\Phi_k \tag{17}$$

where Φ_k is an indicator variable that takes value 1 if agent k is informed about a_k, and 0 otherwise.

It can be easily seen that Ψ_k is invariant to the portfolio decision of the agent from country k, as it is only a function of the portfolio of the buyer, and therefore taken by the seller as given. However, note that Ψ_k does depend on whether or not the seller is informed.

It is useful to follow Lagos and Wright (2005) in defining a function $\ell(q_{z,k,n})$ as the liquidity premium prevailing in a type n meeting with a buyer from country z and a seller from country k. This function represents the increase in the buyer's utility from bringing an additional unit of wealth into the type n meeting over and above the value of just bringing that extra unit of wealth into the next centralized market. $\ell(q_{z,k,n})$ satisfies

$$\ell(q_{z,k,n}) \equiv \frac{\upsilon'(q_{z,k,n})}{\eta'(q_{z,k,n})} - 1. \tag{18}$$

Note that $\ell(q_{z,k,n})$ is only a function of buyer characteristics. Moreover, we also follow Lagos and Wright (2005) in assuming that $\ell'(q_{z,k,n}) \leq 0$, which holds under usual conditions.

Differentiating V_z, the first order conditions for money demand satisfy

$$\frac{\partial V_z}{\partial m_{z,u}} = \phi_u[\lambda_{u,1}\ell(q_{z,u,1}) + 1] \tag{19}$$

and

$$\frac{\partial V_z}{\partial m_{z,r}} = \phi_r[\lambda_{z,r,1}\ell(q_{z,r,1}) + 1]. \tag{20}$$

The first order conditions for asset demand satisfy

$$\frac{\partial V_z}{\partial a_{z,u}} = (\psi_u + \delta_u)[\lambda_{u,2}\ell(q_{z,u,2}) + 1] \tag{21}$$

and

$$\frac{\partial V_z}{\partial a_{z,r}} = (\psi_r + \delta_r)[\lambda_{r,2}\ell(q_{z,r,2}) + 1]. \tag{22}$$

Combining 19, 20, 21, and 22 with the centralized market solution conditions, we obtain solutions for the conditions determining portfolio demand. The demand for currency u satisfies

$$\phi_u \geq \beta\widehat{\phi}_u[\lambda_{u,1}\ell(\widehat{q}_{z,u,1}) + \lambda_{u,2}\ell(\widehat{q}_{z,u,2}) + 1], \tag{23}$$

while the demand for currency r satisfies

$$\phi_r \geq \beta\widehat{\phi}_r[\lambda_{r,1}\ell(\widehat{q}_{z,r,1}) + \lambda_{r,2}\ell(\widehat{q}_{z,r,2}) + 1], \tag{24}$$

where the conditions hold with equality if $\widehat{m_u}$ and $\widehat{m_r}$ are strictly positive, respectively.

The demand for assets satisfy

$$\psi_u \geq \beta(\widehat{\psi}_u + \delta_u)[\lambda_{u,2}\ell(\widehat{q}_{z,u,2}) + 1], \tag{25}$$

and

$$\psi_r \geq \beta(\widehat{\psi}_r + \delta_r)[\lambda_{r,2}\ell(\widehat{q}_{z,r,2}) + 1], \tag{26}$$

where the conditions again hold with equality if $\widehat{a_u}$ and $\widehat{a_r}$ are strictly positive, respectively.

4 Equilibrium

Equilibrium is defined as a solution for asset holdings by agents from u and r, $(m_{u,u}, m_{u,r}, a_{u,u}, a_{u,r})$, and $(m_{r,u}, m_{r,r}, a_{r,u}, a_{r,r})$, asset prices $(\phi_u, \phi_r, \psi_u, \psi_r)$, the terms of trade in the decentralized markets, $(p_k, q_k); (k = u, r)$, and the leisure choices, (x_u, h_u) and (x_r, h_r), which satisfy the maximization conditions of each agent, the bargaining solutions in the decentralized markets, and market clearing in the centralized market.

In the steady state equilibrium, real variables are constant over time, so that $q_z = \widehat{q_z}$, $\phi_z m_z$ and $\psi_z a_z$ are constant, and ϕ_z and M_z grow at a constant rate γ_z $(z = u, r)$. The steady state versions of money demand equations 23 and 24 satisfy

$$\frac{\gamma - \beta}{\beta \lambda_u} \geq (1 - \rho_u)\ell(q_{z,u,1}) + \rho_u \ell(q_{z,u,2}), \tag{27}$$

and

$$\frac{\gamma - \beta}{\beta \lambda_r} \geq (1 - \rho_r)\ell(q_{z,r,1}) + \rho_r \ell(q_{z,r,2}), \tag{28}$$

where the conditions hold with equality for agents that hold strictly positive levels of m_u and m_r respectively.

The demand for assets satisfy

$$\frac{(1 - \beta)\psi_u - \beta \delta_u}{\beta(\psi_u + \delta_u)\lambda_u} = \rho_u \ell(q_{z,u,2}), \tag{29}$$

and

$$\frac{(1-\beta)\psi_r - \beta\delta_r}{\beta(\psi_r + \delta_r)\lambda_r} = \rho_r \ell(q_{z,r,2}), \tag{30}$$

where the conditions hold with equality for agents that hold strictly positive levels of a_u and a_r respectively.

The equilibrium solution is described as the following proposition:

Proposition 1 *There exists a unique steady state monetary equilibrium for which* ($q_{z,u,1}$ *and* $q_{z,u,2}$ *satisfy 27 and 29,* ($q_{z,r,1}$) *and* ($q_{z,r,2}$) *satisfy 28 and 30, prices satisfy* $\phi_k = \eta(q_{z,k,1})/M_{z,k}$ *and* $\psi_k = [\eta(q_{z,k,2} - \eta(q_{z,k,1}]/A_{z,k} - \delta_k$ *where* $(z, k = u, r)$*.*

Proof:

First, we demonstrate that the equilibrium prices are as stated. Consider a type 1 meeting with an agent from country k in which the agent from country z wants to buy $z, k = i, j$. By definition, the buyer can only use country k currency for the purchase in a type 1 meeting. Since the amount of the purchase in a type 1 meeting is equal to $\eta(q_{z,k,1})$ by equation 14, the value of currency holdings in this meeting is $M_{z,k}$ is equal to $\phi_k = \eta(q_{z,k,1})/M_{z,k}$.

Next, consider a type 2 meeting with the same pair of agents. In this meeting, the agent from country k will accept country k assets as well as currency. Since the buyer is illiquid, he uses all of his assets and currency in the transaction. It follows that $\eta(q_{k,z,1})$ of the transaction is financed by currency and $[\eta(q_{z,k,2}) - \eta(q_{z,k,1})]$ is left to be financed from the dividends earned on holdings of asset A_z, $\delta_k A_{z,k}$, as well as the sale of those holdings, valued at $\psi_k A_{z,k}$. It follows that $\delta_k A_{z,k} + \psi_k A_{z,k} = [\eta(q_{z,k,2}) - \eta(q_{z,k,1})]$, which can be solved for ψ_k as stated in Proposition 1.

The existence of an interior solution for ψ_u and ψ_r can be seen from equations 28 and 30. The limit of the left-hand side of either equation as $\psi_k \to \infty$ $(k = u, r)$ is -1, which precludes either equation from holding with equality. Similarly, as $\psi_k \to 0$ the left-hand side of either equation is

∞. Differentiating the left hand sides of 28 and 30 with respect to ψ_k $(k = u, r)$, we obtain

$$\frac{\partial}{\partial \psi_k} = \frac{\lambda_k \delta_k}{[\beta(\psi_k + \delta_k)\lambda_k]^2} \geq 0, (k = u, r) \tag{31}$$

which combined with the fact that the right-hand sides of 28 and 30 are decreasing in ψ_k by inspection (higher asset values raise liquidity, reducing the liquidity premium $\ell(q_{z,k,2})$ $(k = u, r)$) guarantees uniqueness.

5 Comparative statics

Given the equilibrium, we next examine the comparative static impact of a decline in δ_u. First by equation 29, the change in ψ_u with a decline in δ_u satisfies

$$\frac{\partial \psi_u}{\partial \delta_u} = \frac{\delta_u - \beta(\psi_u + \delta_u)\lambda_u \rho_u \ell'(q_{z,u,2})}{\psi_u - \beta(\psi_u + \delta_u)\lambda_u \rho_u \ell'(q_{z,u,2})}. \tag{32}$$

The numerator of equation 32 is unambiguously positive, but the denominator is ambiguous in sign. The necessary condition for $\partial \psi_u / \partial \delta_u \geq 0$ is that $\ell'(q_{z,u,2})$ is not "too large". We require

$$\psi_u \geq \beta(\psi_u + \delta_u)\lambda_u \rho_u \ell'(q_{z,u,2}). \tag{33}$$

In contrast, it can be seen by inspection of equation 30 ψ_r is invariant to a decline in δ_u.

Substituting from equation 29 into equation 27 we obtain

$$\gamma - \beta \geq \beta \lambda_u (1 - \rho_u) \ell(q_{z,u,1}) + \frac{\psi_u}{(\psi_u + \delta_u)}. \tag{34}$$

In the steady state the level of real balances taken by an agent from country z into the decentralized market, $\phi_u m_{z,u}$, will be a constant. However, the steady state value of $\phi_u m_{z,u}$will be endogenous, and in particular a function δ_u. Totally differentiating with respect to $\phi_u m_{z,u}$ and δ_u yields

$$\frac{\partial \phi_u m_{z,u}}{\partial \delta_u} = \frac{\psi_u + \delta_u \frac{\partial \psi_u}{\partial \delta_u}}{(\psi_u + \delta_u)^2 \beta \lambda_u (1-\rho_u) \ell'(q_{z,u,1}) I\{\widehat{\omega}_{z,u,1} < \eta(q^*)\}} \leq 0, \tag{35}$$

as $\frac{\partial \psi_u}{\partial \delta_u}$ can be signed as positive given satisfaction of condition 33.

Again, in contrast, it can be seen by inspection of equation 28, combined with the fact that ψ_r is invariant to a decline in δ_u, that $\phi_r m_{z,r}$ will be invariant to a change in δ_u. This leads to our second proposition:

Proposition 2 *A decline in the payment stream of the risky asset from country u will lead to an appreciation in country u's exchange rate, ϕ_u/ϕ_r.*

Proof:

We have four equations and four unknowns for the price and allocations of country i assets. The four equations are

$$\Lambda_1 \equiv \lambda_{u,u,1} \ell\left(q_{u,u,1}\right) + \lambda_{u,u,2} \ell(q_{u,u,2}) - \frac{1-\beta\gamma_u}{\beta\gamma_u} = 0 \tag{36}$$

$$\Lambda_2 \equiv \lambda_{r,u,1} \ell\left(q_{r,u,1}\right) + \lambda_{r,u,2} \ell\left(q_{r,u,2}\right) - \frac{1-\beta\gamma_r}{\beta\gamma_r} = 0 \tag{37}$$

$$\Lambda_3 \equiv \lambda_{u,u,2} \ell\left(q_{u,u,2}\right) - \frac{\psi_u - \beta\left(\psi_u + \delta_u\right)}{\beta\left(\psi_u + \delta_u\right)} = 0 \tag{38}$$

$$\Lambda_4 \equiv \lambda_{r,u,2}\ell\left(q_{r,u,2}\right) I\left\{\omega_{r,u,2} < \eta\left(q*\right)\right\} - \frac{\psi_u - \beta\left(\psi_u + \delta_u\right)}{\beta\left(\psi_u + \delta_u\right)} = 0 \tag{39}$$

To solve for the comparative static equations, recall that $\omega_{z,k,1} = \varphi_k m_{z,k}$ and $\omega_{z,k,2} = \varphi_k m_{z,k} + \left(\psi_k + \delta_k\right) a_{z,k}$, and

$$\frac{dq}{d\omega} = \frac{1}{\eta'\left(q\right)} = \frac{\left[\theta\upsilon' + \left(1-\theta\right)c'\right]^2}{\theta\left(1-\theta\right)\left(\upsilon - c\right)\left(\upsilon' c" - \upsilon" c'\right) + \theta\left(\upsilon'\right)^2 c' + \left(1-\theta\right)\upsilon'\left(c'\right)^2} \geq 0 \tag{40}$$

Define the following

$$\sigma_{u,1} \equiv \lambda_{u,u,1}\ell'\left(q_{u,u,1}\right)\frac{dq_{u,u,1}}{d\omega} < 0 \tag{41}$$

$$\sigma_{u,2} \equiv \lambda_{u,u,2}\ell'\left(q_{u,u,2}\right)\frac{dq_{u,u,2}}{d\omega} < 0 \tag{42}$$

$$\sigma_{r,1} \equiv \lambda_{r,u,1}\ell'\left(q_{r,u,1}\right)\frac{dq_{r,u,1}}{d\omega} < 0 \tag{43}$$

$$\sigma_{r,2} \equiv \lambda_{r,u,2}\ell'\left(q_{r,u,2}\right)\frac{dq_{r,u,2}}{d\omega} < 0 \tag{44}$$

Then the comparative static equations of the system satisfy

$$\begin{bmatrix} (\sigma_{u1}+\sigma_{u2})\, m_{uu} & (\sigma_{u1}+\sigma_{u2})\,\varphi_u & \sigma_{i2} a_{uu} & \sigma_{u2}\,(\psi_u+\delta_u) \\ (\sigma_{r1}+\sigma_{r2})\,(m_u-m_{uu}) & -(\sigma_{r1}+\sigma_{r2})\,\varphi_u & \sigma_{r2}\,(a_u-a_{uu}) & -\sigma_{r2}\,(\psi_u+\delta_u) \\ \sigma_{u2} m_{uu} & \sigma_{u2}\varphi_u & \sigma_{u2} a_{uu} - \delta_u\beta^{-1}\,(\psi_u+\delta_u)^{-2} & \sigma_{u2}\,(\psi_u+\delta_u) \\ \sigma_{r2}\,(m_u-m_{uu}) & -\sigma_{r2}\varphi_u & \sigma_{r2}\,(a_u-a_{uu}) - \delta_u\beta^{-1}\,(\psi_u+\delta_u)^{-2} & -\sigma_{r2}\,(\psi_i+\delta_i) \end{bmatrix} \tag{45}$$

where

$$\Phi = (\psi_u+\delta_u)\,\varphi_u\sigma_{u2}\sigma_{r2}\,[\sigma_{u1}\sigma_{r2}+\sigma_{u1}\,(\sigma_{r1}+\sigma_{r2})]\, m_u a_u - \tfrac{\varphi_u\delta_u}{\beta(\psi_u+\delta_u)}\,(\sigma_{u1}\sigma_{u2}\,(\sigma_{r1}+\sigma_{r2})+(\sigma_{u1}+\sigma_{u2})\,\sigma_{r1}\sigma_{r2})\, m_u \geq \tag{46}$$

So the determinant is positive

Differentiating Λ_1, Λ_2, Λ_3, and Λ_4, with respect to δ_u yields

$$\frac{\partial\Lambda_1}{\partial\delta_u} = \sigma_{u2} a_{uu} \leq 0 \tag{47}$$

$$\frac{\partial\Lambda_2}{\partial\delta_u} = \sigma_{r2}\,(a_u-a_{uu}) \leq 0 \tag{48}$$

$$\frac{\partial\Lambda_3}{\partial\delta_u} = \sigma_{u2} a_{uu} + \psi_u\beta^{-1}\,(\psi_u+\delta_u)^{-2} \tag{49}$$

$$\frac{\partial\Lambda_4}{\partial\delta_u} = \sigma_{r2}\,(a_u-a_{uu}) + \psi_u\beta^{-1}\,(\psi_u+\delta_u)^{-2} \tag{50}$$

To calculate $\partial\varphi_u/\delta_u$, the numerator matrix satisfies

$$\begin{bmatrix} -\sigma_{u2}a_{uu} & (\sigma_{u1}+\sigma_{u2})\,\varphi_u & \sigma_{u2}a_{uu} & \sigma_{u2}(\psi_u+\delta_u) \\ -\sigma_{r2}(a_u-a_{uu}) & -(\sigma_{r1}+\sigma_{r2})\,\varphi_u & \sigma_{r2}(a_u-a_{uu}) & -\sigma_{r2}(\psi_u+\delta_u) \\ -\sigma_{u2}a_{uu}-\psi_u\beta^{-1}(\psi_u+\delta_u)^{-2} & \sigma_{u2}\varphi_u & \sigma_{u2}a_{uu}-\delta_u\beta^{-1}(\psi_u+\delta_u)^{-2} & \sigma_{u2}(\psi_u+\delta_u) \\ -\sigma_{r2}(a_u-a_{uu})-\psi_u\beta^{-1}(\psi_u+\delta_u)^{-2} & -\sigma_{r2}\varphi_u & \sigma_{r2}(a_u-a_{uu})-\delta_u\beta^{-1}(\psi_u+\delta_u)^{-2} & -\sigma_{r2}(\psi_u+\delta_u) \end{bmatrix} \tag{51}$$

The determinant of this matrix satisfies

$$\Phi = (\sigma_{u1}+\sigma_{r1})\,\sigma_{u2}\sigma_{r2}\varphi_u\beta^{-1}a_u \leq 0 \tag{52}$$

So by Cramer's rule, the comparative statics satisfy

$$\frac{\partial\varphi_u}{\partial\delta_u} = \frac{(\sigma_{u1}+\sigma_{r1})\,\sigma_{u2}\sigma_{r2}(\psi_u+\delta_u)\,a_u}{m_u\left[\beta(\psi_u+\delta_u)^2\sigma_{u1}\sigma_{u2}\sigma_{r2}(\sigma_{r1}+2\sigma_{r2})\,a_u-\delta_u\left(\sigma_{u1}\sigma_{u2}(\sigma_{r1}+\sigma_{r2})+(\sigma_{u1}+\sigma_{u2})\,\sigma_{r1}\sigma_{r2}\right)\right]} \leq 0 \tag{53}$$

as stated in Proposition 2.

6 Impact of Central Bank Liquidity Injections

We next turn to the predicted impact of the central bank auctions. We consider the capital injections as analogous to an increase in $m_{r,u}$ in the decentralized market. In other words, one can consider the injections as occurring subsequent to the fall in δ_u. As was the case empirically, the capital injections are assumed to be loans. For tractability, we assume that these swaps are to be repaid before the next period's entry into the centralized market.

The impact of the liquidity injection on an agent from a foreign country can then be represented in terms of the change in the decentralized market value function with an increase in US currency holdings, shown in equation 19. Differentiating $\partial V_r/\partial m_{r,u}$ with respect to $\lambda_{r,u}$ yields

$$\frac{\partial^2 V_r}{\partial m_{r,u}\partial\lambda_{r,u}} = \phi_u[(1-\rho_u)\ell(q_{r,u,1})I_{r,u,1} + \rho_u\ell(q_{r,u,2})I_{r,u,2}] \geq 0. \tag{54}$$

Similarly, differentiating $\partial V_r/\partial m_{r,u}$ with respect to ρ_u, yields

$$\frac{\partial^2 V_r}{\partial m_{r,u}\partial\rho_u} = \phi_u\lambda_{r,u}[-\ell(q_{r,u,1})I_{r,u,1} + \ell(q_{r,u,2})I_{r,u,2}] < 0. \tag{55}$$

since $\ell(q_{r,u,1}) \leq \ell(q_{r,u,2})$.

Finally, differentiating with respect to $\ell(q_{r,u,1})$ yields

$$\frac{\partial^2 V_r}{\partial m_{r,u}\partial\ell(q_{r,u,1})} = \theta_u\lambda_{r,u,1}I_{r,u,1} > 0. \tag{56}$$

the solution for $\ell(q_{r,u,2})$ is similar.

These results imply three characteristics for sensitivity to the auctions. First, the benefits are increasing in $\lambda_{r,u}$, which indicates the probability of needing to transact in US dollars in the decentralized market. Second, the benefits of the capital injections are increasing in $-\rho_u$, the probability of being paired with an uninformed agent. This term can also be interpreted loosely as representing asset opaqueness. Lastly, the impact is increasing in the liquidity premium, $\ell(q_{r,u,n}); n = 1, 2$, or decreasing in the liquidity position of the country.

References

Lagos, R., and G. Rocheteau (2008): "Money and Capital as Competing Media of Exchange," *Journal Economic Theory*, 142(1), 247–258.

Lagos, R., and R. Wright (2005): "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy*, 113(3), 463–484.

Lester, B., A. Postlewaite, and R. Wright (2009): "Information, Liquidity, Asst Prices and Monetary Policy," manuscript, University of Pennsylvania.